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Casimir–Polder forces, boundary conditions and fluctuations

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Abstract

We review different aspects of atom–atom and atom–wall Casimir–Polder forces. We first discuss the role of a boundary condition on the interatomic Casimir–Polder potential between two ground-state atoms, and give a physically transparent interpretation of the results in terms of vacuum fluctuations and image atomic dipoles. We then discuss the known atom–wall Casimir–Polder force for ground- and excited-state atoms, using a different method which is also suited to extension to time-dependent situations. Finally, we consider the fluctuation of the Casimir–Polder force between a ground-state atom and a conducting wall, and discuss possible observation of this force fluctuation.

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1. Introduction

Casimir–Polder forces are long-range interactions between atoms or molecules, or between atoms/molecules and neutral macroscopic objects, due to their interaction with the electromagnetic radiation field in the vacuum state [1]. In the case of two ground-state atoms the Casimir–Polder potential energy behaves as R^{-6} for interatomic distances smaller than typical atomic transition wavelenghts from the ground state (near zone) and as R^{-7} for larger distances (far zone) [2, 3]. In the case of a ground-state atom and an infinite conducting wall, the potential energy decreases like d^{-3} in the near zone and d^{-4} in the far zone, with d being the atom–wall distance. The atom–wall Casimir–Polder force, although very tiny, has been recently measured with precision, both in the near and in the far zone [4–8]. The atom–atom van der Waals/Casimir–Polder energy is still weaker, but experimental indirect evidences of them have existed for a long time, in agreement with theoretical predictions [9]. Direct measurements of the retarded van der Waals attraction in mesoscopic systems have also been obtained [10, 11]. Possible enhancement of such interactions due to the presence of boundary conditions and resonance effects has been recently suggested [12–14].

In this paper we review some results, and relative physical interpretation, about the atom–atom Casimir–Polder force when boundary conditions are present, and about the atom–wall Casimir–Polder force. In section 2, we discuss the force between two ground-state atoms when a perfectly conducting plate is present. Section 3 is devoted to the well-known Casimir–Polder force between a ground- or excited-state atom and a conducting wall, obtained from a different point of view which can also be directly extended to dynamical situations. Finally, in section 4 we discuss quantum fluctuations of the atom–wall Casimir–Polder force, their possible measurement and an estimate of these fluctuations in actual experiments.

2. Effect of a boundary condition on the interatomic van der Waals/Casimir–Polder force

In this section we calculate the Casimir–Polder potential between two neutral ground-state atoms placed in front of a perfectly conducting infinite plate. For our purposes, it is convenient to use an effective Hamiltonian obtained from the usual forms of the interaction Hamiltonian in the dipole approximation by a unitary transformation [15]. For two atoms (A and B) interacting with the radiation field, this effective Hamiltonian is

$$H_I = -\frac{1}{2}\sum_{i=A,B}\sum_{\mathbf{k}\mathbf{k}'jj'}\alpha_i(\omega_k)\mathbf{E}_{\mathbf{k}j}(\mathbf{r}_i)\cdot\mathbf{E}_{\mathbf{k}'j'}(\mathbf{r}_i) \tag{1}$$

where $\mathbf{r}_i$ is the position of atom i, $\alpha_i(\omega_k)$ its dynamical polarizability (assumed isotropic), and

$$\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{k}j}\mathbf{E}_{\mathbf{k}j}(\mathbf{r}) = \mathrm{i}\sum_{\mathbf{k}j}\sqrt{\frac{2\pi\hbar\omega_k}{V}}\mathbf{f}(\mathbf{k}j,\mathbf{r})\big(a_{\mathbf{k}j} - a^{\dagger}_{\mathbf{k}j}\big) \tag{2}$$

is the transverse displacement field operator, $\mathbf{f}(\mathbf{k}j,\mathbf{r})$ are appropriate mode functions taking into account the boundary conditions and j is a polarization index. The effective Hamiltonian (1) is correct up to the second order in the electron charge e. Our calculation proceeds in two steps. We first obtain the dressed ground state of one atom (A), and then we evaluate the effective interaction energy of the second atom (B) with the field fluctuations generated by the virtual photon cloud dressing the first atom. The dressed ground state of atom A at the lowest significant order in the atom–radiation coupling is obtained by straightforward perturbation theory

$$|\{0_{\mathbf{k}j}\},\downarrow\rangle_D = |\{0_{\mathbf{k}j}\},\downarrow\rangle - \frac{\pi}{V}\sum_{\mathbf{k}\mathbf{k}'jj'}\alpha_{\mathrm{A}}(\omega_k)\frac{\sqrt{kk'}}{k+k'}\mathbf{f}(\mathbf{k}j,\mathbf{r}_{\mathrm{A}})\cdot\mathbf{f}(\mathbf{k}'j',\mathbf{r}_{\mathrm{A}})|1_{\mathbf{k}j}1_{\mathbf{k}'j'},\downarrow\rangle \tag{3}$$

where $|\downarrow\rangle$ denotes the ground state of atom A. The interaction energy is obtained from the average value of the effective interaction Hamiltonian of atom B with the field (the term with $i = B$ of equation (1)) on the dressed ground state (3) of atom A, that is

$$\Delta E_{\mathrm{AB}} = -\frac{1}{2}\sum_{\mathbf{k}\mathbf{k}'jj'}\alpha_{\mathrm{B}}(\omega_k)_D\langle\{0_{\mathbf{k}j}\},\downarrow|\mathbf{E}_{\mathbf{k}j}(\mathbf{r}_{\mathrm{B}})\cdot\mathbf{E}_{\mathbf{k}'j'}(\mathbf{r}_{\mathrm{B}})|\{0_{\mathbf{k}j}\},\downarrow\rangle_D. \tag{4}$$

After some lengthy algebra, in the far zone we obtain

$$\begin{aligned}\Delta E_{\mathrm{AB}}(R,\bar{R}) = {} & -\frac{23}{4\pi}\hbar c\frac{\alpha_{\mathrm{A}}(0)\alpha_{\mathrm{B}}(0)}{R^7} - \frac{23}{4\pi}\hbar c\frac{\alpha_{\mathrm{A}}(0)\alpha_{\mathrm{B}}(0)}{\bar{R}^7}\\ & + \frac{8}{\pi}\hbar c\frac{\alpha_{\mathrm{A}}(0)\alpha_{\mathrm{B}}(0)}{R^3\bar{R}^3(R+\bar{R})^5}[R^4\sin^2\vartheta + 5R^3\bar{R}\sin^2\vartheta\\ & + R^2\bar{R}^2(6+\sin^2\vartheta+\sin^2\bar{\vartheta}) + 5R\bar{R}^3\sin^2\bar{\vartheta} + \bar{R}^4\sin^2\bar{\vartheta}]\end{aligned} \tag{5}$$

where $\alpha_i(0)$ are the static atomic polarizabilities, R is the distance between the atoms, $\bar{R}$ is the distance between one atom and the image of the other atom reflected on the wall, and ϑ ($\bar{\vartheta}$) is the angle between $\mathbf{R}$ ($\bar{\mathbf{R}}$) and the normal to the plate [16, 17].

Equation (5) is a general expression of the Casimir–Polder potential energy between the two atoms in the presence of the conducting plate in the far zone. We note in (5) the presence of three terms: the usual R^{-7} Casimir–Polder potential between the two atoms (as in the absence of the plate), the $\bar{R}^{-7}$ Casimir–Polder-like interaction between one atom and the image of the other atom reflected on the plate, and a term depending on both variables R and $\bar{R}$. It is possible to show that the interatomic force is attractive for any spatial configuration of the system. Although expression (5) has been already obtained by fourth-order perturbation theory [16], we have now obtained it by a different and simpler method, which stresses the role of dressed field fluctuations modified by the presence of the conducting plate. In fact, we have shown that the Casimir–Polder potential can be seen as the interaction energy of one atom with the photon cloud dressing the other atom, which has been modified by the presence of the conducting plate.

Potential (5) can also be obtained with a different physical model, based on the properties of the spatial correlations of vacuum fluctuations which are modified by the conducting plate [18, 19]. In fact, the zero-point electromagnetic field induces a fluctuating dipole moment on each atom which is related to the fluctuating zero-point field by (assuming an isotropic atom)

$$\mu_\ell(\mathbf{k}j) = \alpha(\omega_k) E_\ell(\mathbf{k}j, \mathbf{r}). \tag{6}$$

Because vacuum fluctuations are spatially correlated, the induced dipole moments are correlated too. We now let the two induced dipoles interact through the classical potential between oscillating dipoles [17]

$$\begin{aligned} V_{lm}(k, R, \bar{R}) &= V_{lm}(k, R) - \sigma_{lp} V_{pm}(k, \bar{R}) \\ &= (\nabla^2 \delta_{lm} - \nabla_l \nabla_m)^R \frac{\cos(kR)}{R} - \sigma_{lp} (\nabla^2 \delta_{pm} - \nabla_p \nabla_m)^{\bar{R}} \frac{\cos(k\bar{R})}{\bar{R}} \end{aligned} \tag{7}$$

where the first term is the interaction between the two induced dipole moments as in the absence of the plate, and the second term takes into account the presence of the conducting plate with a contribution due to the interaction between one atom and the image of the other atom. In (7), which is indeed related to the real part of the field Green function as obtained in [20] for our planar geometry, σ_{lp} is the reflection matrix with respect to the plate (for a non-perfectly conducting plate, it would also depend on the wave vector), and the superscripts R, $\bar{R}$ indicate the variables with respect to which derivatives are taken. The interaction energy between the two atoms is thus

$$\begin{aligned} V_{AB} &= \sum_{\mathbf{k}j} \langle 0 | \mu_\ell^A(\mathbf{k}j) \mu_m^B(\mathbf{k}j) | 0 \rangle V_{\ell m}(k, R, \bar{R}) \\ &= \sum_{\mathbf{k}j} \alpha_A(\omega_k) \alpha_B(\omega_k) \langle 0 | E_\ell(\mathbf{k}j; \mathbf{r}_A) E_m(\mathbf{k}j; \mathbf{r}_B) | 0 \rangle V_{\ell m}(k, R, \bar{R}) \end{aligned} \tag{8}$$

where the spatial correlation function of the electric field operator appears, which, similarly to V_{lm}, depends on the boundary conditions. After straightforward algebra, evaluation of (8) again yields equation (5). This gives a second physically transparent interpretation of the origin of the Casimir–Polder energy between the two atoms with boundary conditions present. It also gives support to the phenomenological model used and indicates that it could also be applied to more complicated boundary conditions. The effect of boundary conditions on the Casimir–Polder interaction between atoms, when approximated as harmonic oscillators, has also been considered in the literature at zero and finite temperatures using Green function

techniques [21, 22]. An interpretation of the van der Waals potential in terms of dipole images has also been given in [23].

3. The atom–wall Casimir–Polder force for ground- or excited-state atoms

We now consider the Casimir–Polder force between one atom and a neutral perfectly conducting infinite plate. The atom–plate Casimir–Polder interaction is well known in the literature, both for ground- and excited-state atoms [24, 25]; we shall however derive it from a different point of view. In the multipolar coupling scheme, the atom–field interaction is given by $H_I = -\boldsymbol{\mu}\cdot\mathbf{E}(\mathbf{r})$, where $\boldsymbol{\mu}$ and $\mathbf{E}(\mathbf{r})$ are respectively the atomic dipole moment and the transverse displacement field, $\mathbf{r}$ being the position of the atom (we do not use the effective Hamiltonian (1) because it is limited to systems in their ground state). We consider a two-level atom with transition frequency ck_0 between the two levels 1 and 2, interacting with the field by the multipolar coupling Hamiltonian. Due to the atom–field interaction, the unperturbed states become dressed; in our approach, the atom–plate Casimir–Polder interaction is obtained from the average value of H_I on the first-order dressed state $|\{0_{\mathbf{k}j}\}\downarrow(\uparrow)\rangle_D$ as [26]

$$\Delta E_{g(e)} = \frac{1}{2}{}_D\langle\{0_{\mathbf{k}j}\},\downarrow(\uparrow)|H_I|\{0_{\mathbf{k}j}\},\downarrow(\uparrow)\rangle_D \tag{9}$$

where $\downarrow$ and $\uparrow$ refer respectively to a ground-state and an excited-state atom. The interaction energy is then

$$\Delta E_{g(e)} = -\frac{4\pi}{V}\sum_{\mathbf{k}j}\frac{k}{k\pm k_0}\left(\boldsymbol{\mu}_{21}\cdot\mathbf{f}(\mathbf{k}j,\mathbf{r})\right)^2 \tag{10}$$

where $\mathbf{f}(\mathbf{k}j,\mathbf{r})$ are the field mode functions evaluated at the position of the atom; the plus sign refers to a ground-state atom and the minus sign to an excited-state atom (the known difficulties of the two-level model in dealing with frequency shifts near an interface [27] do not seem relevant in our case; our results, derived in the multipolar coupling scheme, can be straightforwardly generalized to the case of a multilevel atom, obtaining an expression analogous to (10) with a summation over all atomic states). For isotropic atoms, after summation over the orientations of the atomic dipole, the expressions for the Casimir–Polder force for a ground- and an excited-state atom are

$$F_g = -\frac{\mu^2}{12\pi d^4}\left[8k_0d - 6\left(2k_0^2d^2-1\right)f(2k_0d) - 4k_0d\left(2k_0^2d^2-3\right)g(2k_0d)\right] \tag{11}$$

$$\begin{aligned}F_e = \frac{\mu^2}{12\pi d^4}\Big[&8k_0d - 6\left(2k_0^2d^2-1\right)\left(f(2k_0d)-\pi\cos(2k_0d)\right)\\ &- 4k_0d\left(2k_0^2d^2-3\right)\left(g(2k_0d)-\pi\sin(2k_0d)\right)\Big]\end{aligned} \tag{12}$$

where $f(z)$ and $g(z)$ are the auxiliary functions of the sine- and cosine-integral functions [28]. The energy shifts in (10) can also be expressed in terms of the field Green's functions, which include the summation over the field modes [20, 29]; also, the contributions from vacuum fluctuations and radiation reaction (which includes the dipole image field) can be explicitly separated [30]. For a ground-state atom, equation (11) gives an attractive force for any atom–wall distance. For an excited-state atom, equation (12) shows spatial oscillations of the force with a periodicity related to k_0, yielding spatial regions where the force is attractive and regions where the force is repulsive. Although these results are already known in the literature (see e.g. [24, 31]), our approach gives a physical picture of the atom–wall interaction as the average interaction energy of the atom with the field fluctuations associated with its dressing photon

cloud, whose properties depend on the presence of the wall. Also, this approach allows a conceptually easy evaluation of dynamical, i.e. time-dependent, atom–plate Casimir–Polder forces in nonequilibrium situations such as during the dynamical self-dressing of the atom [32]. A similar approach has been already used for the dynamical Casimir–Polder force between two atoms [33]; time dependence of the force for an atom in a superposition of energy states has also been recently considered [34].

4. Fluctuations of the atom–wall Casimir–Polder force

The atom–wall Casimir–Polder force in section 3 is an average value of the force, and quantum fluctuations are expected. Atom–wall force fluctuations have been already considered (in the far zone only) by a Langevin-like equation for the position of the atom [35]. In this section we evaluate the atom–wall Casimir–Polder force fluctuation for a ground-state atom by using an operator directly associated with the force acting on the atom; our method can be used both in the near and in the far zone. The atom–radiation coupling is described using the effective Hamiltonian (1). The first step is to define a quantum operator directly associated with the force experienced by the atom, in order to easily calculate its fluctuation. This operator is defined by $F = -\partial H_I/\partial d$, i.e. by taking minus the formal derivative of the operator H_I with respect to the atom–wall distance d, inspired by a classical quasi-static analogy (we neglect the atomic translational degrees of freedom). The mean value of this operator on the ground state of our system gives back the correct value of the force of the previous section

$$F(d) = -\frac{3\hbar c\alpha}{2\pi d^5} \tag{13}$$

(far zone) where d is the atom–wall distance and $\alpha = 2\mu_{21}^2/3\hbar\omega_0$ the static polarizability of the atom. Once we have a force operator, we can square it in order to obtain the force fluctuation $\Delta F = (\langle F^2\rangle - \langle F\rangle^2)^{1/2}$, where average values are taken on the ground state of the system. Unfortunately, a direct evaluation of ΔF yields a result containing non-regularizable ultraviolet divergences. Similar divergences were obtained for the fluctuation of the macroscopic Casimir force between two parallel infinite conducting plates. In order to solve this problem we have followed the approach initially proposed by Barton [36–38], by taking into account the fact that any force measurement has a finite duration and thus introducing a time-averaged force operator. This operator is an average of the force operator in the Heisenberg representation over a weight function $f(t)$ describing the instrumental response, that is

$$\overline{F} = \int_{-\infty}^{+\infty} \mathrm{d}t\; f(t)\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}Ht} F\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht} \tag{14}$$

where F is the force operator defined above and H is the Hamiltonian of the system. This time-averaging procedure introduces a cutoff in the frequency integration related to the duration of the measure, which makes the force fluctuation finite. Using this technique and choosing a Lorentzian of width T as the instrumental response function $f(t) = \frac{1}{\pi}\frac{T}{t^2+T^2}$, we obtain

$$\Delta\overline{F}^2 = \frac{1}{2}\sum_{\mathbf{k}\mathbf{k}'jj'} |\langle 0|F|1_{\mathbf{k}j}1_{\mathbf{k}'j'}\rangle|^2 |g(\omega_k + \omega_{k'})|^2 \tag{15}$$

where $g(\omega) = \exp(-\omega T)$ is the Fourier transform of $f(t)$. It is thus evident that the use of the time-averaged operator $\overline{F}$ naturally introduces a frequency cutoff, making the force fluctuation finite. This is physically reasonable since an instrument characterized by a response time T is not able to resolve processes with frequencies higher than T^{-1}. By considering the relative fluctuation of the force (ratio between $\Delta\overline{F}$ and the mean force), we find two different regimes

according to the ratio between the atom–wall distance d and cT. When $d \ll cT$ (long measurements), we get

$$\frac{\Delta\overline{F}}{|\langle 0|F|0\rangle|} \sim \left(\frac{d}{cT}\right)^6 \tag{16}$$

while when $d \gg cT$ (short measurements), we obtain

$$\frac{\Delta\overline{F}}{|\langle 0|F|0\rangle|} \sim \left(\frac{d}{cT}\right)^5. \tag{17}$$

In the former case, force fluctuations are hardly observable; in the latter case, the force largely fluctuates around its mean value. In order to analyze the observability of the fluctuations, we can consider typical parameters from actual experimental setups for the atom–wall force measurements as in [4, 5]. In this case we have $d \sim 1\ \mu$m; hence the duration T distinguishing the two regimes is $\sim 10^{-14}$ s [39]. Therefore, observation of the quantum fluctuations of the Casimir–Polder force requires very short measurements compared to measurement times in recent experiments (for example, a rough estimate in the experiments in [4, 5] gives $T \sim 10^{-5}$ s, much longer than the time required to observe the force fluctuation). Observation of the fluctuations of the Casimir–Polder force requires much shorter measurement times, or to envisage situations where the force and its fluctuations could be significantly enhanced (for example through resonance effects for atoms in excited states).

5. Conclusions

We have reviewed several different aspects of Casimir–Polder forces. In section 2 we have considered how boundary conditions can affect the Casimir–Polder interaction between two ground-state atoms, and given a transparent physical interpretation of the results in terms of zero-point fluctuations and image dipoles. In section 3 we have considered the Casimir–Polder force between an atom and a conducting wall, obtained as the interaction energy of the atom with its dressing field fluctuations. Finally, in section 4 we have analyzed quantum fluctuations of the Casimir–Polder force between a ground-state atom and a conducting wall, and discussed their observability.

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